Josephson current through a long quantum wire

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The dc Josephson current through a long SNS junction receives contributions from both Andreev bound states localized in the normal region as well as from scattering states incoming from the superconducting leads. We show that in the limit of a long junction, this current, at low temperatures, can be expressed entirely in terms of properties of the Andreev bound states at the Fermi energy: the normal and Andreev reflection amplitudes. This has important implications for treating interactions in such systems.

As was shown by Josephson [1] a current can pass between two superconductors separated by a normal material, even with zero potential difference. At temperature T, this Josephson current is determined by the difference of the phase of the order parameter in the two superconductors, χ:

$$I[\chi;T] = 2e\frac{dF}{d\chi} \qquad (1)$$

where F is the free energy [2]. Using the Bardeen-Cooper-Schrieffer (BCS) approximation in the superconducting leads and ignoring interactions in the normal region, $I(\chi)$ can be expressed as a sum over single quasiparticle energy levels, E_n:

$$I[\chi;T] = 2e\sum_n f(E_n)\frac{dE_n}{d\chi} \qquad (2)$$

where $f(E) = 1/[e^{E/T}+1]$ is the Fermi function and we measure energies from the chemical potential. In general, these states are of at least two distinct forms. There are Andreev bound states (ABS) [3], with energies $|E| < \Delta$, where Δ is the superconducting gap, which are localized in the normal region and whose wavefunctions decay exponentially into the superconducting leads. There are also scattering states (SS), with energies $|E| > \Delta$ corresponding to waves coming in from infinity in the superconducting leads and being reflected and transmitted. If the bottom of the band in the normal material is lower than the bottom of the band in the superconducting leads, there are, in addition, normal bound states, localized in the normal region which also decay exponentially in the leads. In general, all types of states contribute to the Josephson current.

The limit of a long narrow normal region was considered in [4], using a nearest neighbor tight-binding model and initially ignoring interactions. In particular, it was assumed that the length of the normal region, ℓ, was much greater than the coherence length, or equivalently than the finite size gap, $\pi v_F/\ell \ll \Delta$ where v_F is the Fermi velocity in the normal region. Furthermore, only $T=0$ was considered. In this limit it appears natural to integrate out the gapped superconductors and derive an effective Hamiltonian for the normal region, with local pairing interactions induced by the proximity effect at its boundaries. Such as effective Hamiltonian was used to derive the Josephson current. In this approach, only ABS's are considered. Due to a remarkable cancellation between pairs of ABS's it was found that the current, to order $1/\ell$, could be expressed in terms of scattering amplitudes at the Fermi energy only.

This approach was called into question by the results of [5]. There it was verified that the ABS's gave the entire Josephson current for long junctions in the unrealistic limit $\Delta > 2J$ where $4J$ is the bandwidth in the normal region. However, numerical results for intermediate length junctions seemed to suggest a significant contribution from SS's for $\Delta/(2J) < 1$.

In this letter we study general models of long non-interacting SNS junctions without integrating out the superconducting leads. We prove that, for v_F/ℓ and $T \ll \Delta$, the Josephson current can indeed be expressed in terms of data at the Fermi level only. We emphasize that states far from the Fermi energy make large contributions to the current; it is just that these nearly cancel for large ℓ, leading to our result.

This finding is important because integrating out the superconductors provides a powerful method for including interaction effects in the normal region, based on boundary conformal field theory techniques [4]. (See also [6].) While [4, 6] only considered the dc Josephson current, the techniques introduced there can be extended [7] to the ac case by allowing for the phase of the boundary pairing interactions to evolve linearly in time, $\chi = eVt$, where V is the voltage difference. A possible experimental realization of such a long SNS junction might be provided by a carbon nanotube between bulk superconductors. Using $v_F \approx 8.1\times10^5$ m/s, $\pi v_F/\ell \approx .5$ meV for $\ell=$ 3 microns. Thus, obtaining sufficiently long clean nanotubes coupled to sufficiently high T_c superconductors to satisfy $\pi v_F/\ell \ll \Delta$ may be near the limits of current nanotechnology.

Our proof is based on expressing the total current, summed over all types of states, as a contour integral in

the complex energy plane, involving the S-matrix. In particular, we derive this contour integral for the $T = 0$ case and observe, using analytic properties of the S-matrix, that the contour can be deformed into an integral along the entire imaginary E-axis. We then show that, in the large ℓ limit, $v_F/\ell, \ll \Delta$, the integral is determined by the S-matrix at the Fermi energy, and apply our general result to the "Blonder-Tinkham-Klapwijk (BTK) model", obtaining an explicit formula for the current for $v_F/\ell \ll \Delta$. We extend our contour methods to finite T, by expressing the resulting current in terms of a sum along the imaginary energy axis at the Matsubara frequencies, $E = i\omega_n \equiv i2\pi(n + 1/2)T$. We find that the current vanishes exponentially when $T \gg v_F/\ell$. Finally, we mention that our approach may readily be extended to tight-binding models, such as the one discussed in [4], whose results for the current we recover when $T = 0$ and $v_F/\ell \ll \Delta$.

We consider a general SNS model in the non-interacting, BCS approximation with a gap function $\Delta(x)$ of magnitude Δ at $|x| \to \infty$ and 0 in the central region. We also include a normal potential, $V(x)$ which vanishes at $|x| \to \infty$. A 4×4 transmission matrix, M may be defined which relates the asymptotic wave-function in the S regions at $x \to \pm\infty$, $\vec{A}^+ = M\vec{A}^-$ with Bogoliubov-DeGennes wave-function obeying (for $(x \to \pm\infty)$, respectively):

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} \to \begin{bmatrix} \cos(\Psi/2) \\ -e^{\pm i\chi/2}\sin(\Psi/2) \end{bmatrix}_p \left[A_1^\pm e^{i\beta_p x} + A_2^\pm e^{-i\beta_p x}\right]$$
$$+ \begin{bmatrix} -e^{\mp i\chi/2}\sin(\Psi/2) \\ \cos(\Psi/2) \end{bmatrix}_h \left[A_3^\pm e^{-i\beta_h x} + A_4^\pm e^{i\beta_h x}\right] \;, \tag{3}$$

where the particle and hole momenta, for energy E, are $\beta_{p/h}^2 = 2m_S\{\mu \pm (E^2 - \Delta^2)^{\frac{1}{2}}\}$, with m_S the electron effective mass in the S regions and μ the chemical potential, and $\Psi \equiv -\arcsin(\Delta/E)$. (For simplicity, we take an energy-independent gap, Δ, but our results can be extended to more realistic models.) Eq. (3) applies also to the ABS regime, in which $E^2 - \Delta^2 < 0$. In this case, the phases of the arguments of the complex square root functions are always chosen so that $\mathrm{Im}(\beta_p) \geq 0$ and $\mathrm{Im}(\beta_h) \leq 0$ [8]. The S-matrix, which expresses outgoing waves (A_1^+, A_3^+, A_2^-, A_4^-) in terms of incoming waves can be expressed in terms of M. In particular:

$$\det[S] = \frac{M_{1,1}M_{3,3} - M_{1,3}M_{3,1}}{M_{2,2}M_{4.4} - M_{2,4}M_{4,2}} \equiv \frac{\mathcal{F}(E;\chi)}{\mathcal{G}(E;\chi)} \quad . \tag{4}$$

In Eq. (4), $\mathcal{F}(E;\chi)$ and $\mathcal{G}(E;\chi)$ must be regarded as functions of E in the complex E-plane. We chose them to obey several convenient properties, which are crucial for our derivation:

i) They are always finite for finite E. This can be easily achieved by shifting poles of $\mathcal{G}$ into zeroes of $\mathcal{F}$ and vice versa;

ii) They have no common zeroes. [Possible common zeroes (e.g. E_0), could always be cancelled by a redefinition: $\mathcal{F}(E;\chi) \to \mathcal{F}(E;\chi)/(E - E_0)$, $\mathcal{G}(E;\chi) \to \mathcal{G}(E;\chi)/(E - E_0)$, without changing Eq. (4)].

iii) $\mathcal{F}(E;\chi) = \mathcal{G}^*(E;\chi)$. Here this equation refers to complex conjugating *the function* without complex conjugating its argument, E. This condition is consistent with the requirement that $|\det[S]| = 1$ for scattering states.

iv) $\mathcal{G}(E;\chi)$ can be defined to have branch cuts along the real E-axis, corresponding to the nonzero density of scattering states in the leads. This is due to the fact that $\mathcal{G}(E;\chi)$ depends on E via β_p and β_h and that they become double-valued functions of E, for $|E| > \Delta$.

v) $\partial_\chi \ln \mathcal{G}(E;\chi)$ vanishes rapidily at $|E| \to \infty$ along any ray not parallel to the real axis. This condition is crucial to allow for pertinently deforming the integration path in the energy plane, when computing $I^{(0)}[\chi]$.

vi) $\mathcal{G}(E;\chi)$ is real in the bound state region: the real axis with $-\Delta \leq E \leq \Delta$.

These conditions appear to determine $\mathcal{F}(E;\chi)$ and $\mathcal{G}(E;\chi)$ uniquely except for an overall multiplicative constant factor. Moreover, they imply that zeroes of $\mathcal{G}(E;\chi)$ correspond to poles of $\det[S]$. These conditions imply that there are no poles of $\det[S]$ off the real axis. We are actually dealing with a 2-sheeted Riemann surface, due to the branch cuts. We may regard $\mathcal{F}(E;\chi)$ as being $\mathcal{G}(E;\chi)$ on the second sheet of the Riemann surface. With the definition we gave of $\beta_p(E), \beta_h(E)$, the zeroes of $\mathcal{G}(E;\chi)$, corresponding to poles of $\det[S]$, occur either on the real axis, or else off-axis on the second sheet of the Riemann surface. This property of $\det[S]$ follows from general principles. Since the S-matrix can be derived from the retarded Green's function it should have no singularities in the upper half plane. Since

$$\mathcal{G}(E;\chi) = \mathcal{G}^*(E;\chi) \tag{5}$$

in the BS region, we can use the Schwartz reflection principle to define its unique analytic continuation to the entire first sheet of the Riemann surface, where it obeys Eq. (5). Thus if $\mathcal{G}(E;\chi)$ had a zero in the lower half-plane, at E_0, it would have to have a twin at energy E_0^* in the upper half-plane. This would violate this basic property of S telling us that no such zeroes exist.

The ABS's correspond to poles of the S-matrix and therefore to the zeroes of $\mathcal{G}(E)$. This allows us to write the contribution to the ground state energy from ABS's as:

$$E_{ABS}^{(0)} = -\frac{1}{2\pi i}\oint_{\Gamma_{ABS}} dE \ln \mathcal{G}(\mathcal{E};\chi) \tag{6}$$

where the contour Γ_{ABS} in the complex energy plane surrounds the negative energy ABS's. In the SS energy regime, for $\Delta < |E| < \sqrt{\mu^2 + \Delta^2}$ (SS1 energy regime), the S-matrix is unitary and $\mathcal{F} = \mathcal{G}^*$. $\det S(E)$ can be written in terms of the particle and hole phase shifts for the 2 scattering solutions at each energy, σ_p^a, σ_h^a, respec-

tively:

$$\det S = \prod_{a=1}^{2} \exp[i(\sigma_p^a - \sigma_h^a)]. \tag{7}$$

On the other hand, for $|E| > \sqrt{\mu^2 + \Delta^2}$ (SS2 energy regime), only particles propagate, but we find Eq. (7) remains true with σ_h^a set to zero. In this way, we obtain the total contribution to the ground state energy from scattering states in the form:

$$E_{SS}^{(0)} = \epsilon_{SS}^0 - \frac{1}{2\pi i}\int_{-\infty}^{-\Delta} dE \ln\left[\frac{\mathcal{G}^*(E;\chi)}{\mathcal{G}(E;\chi)}\right] \tag{8}$$

where ϵ_{SS}^0 is independent of χ. Using Eq. (5), the second term in Eq. (8) can be also be written as a contour integral in the complex energy plane, like Eq. (6), with the contour now running on both sides of the branch cut in $\mathcal{G}$ along the real E-axis from $-\infty$ to $-\Delta$. These two terms can be combined, allowing us to write a simple unified formula for the zero-temperature Josephson current:

$$I^{(0)}[\chi] = -\frac{2e}{2\pi i}\int_\Gamma dE\, \partial_\chi\{\ln \mathcal{G}(E;\chi)\} \tag{9}$$

where the contour Γ runs infinitesimally above and below the negative E axis. Due to the properties of $\mathcal{G}(E;\chi)$ discussed above, Γ can be deformed into a single line running along the imaginary E-axis from $-\infty$ to ∞. [See Fig. (1).]

We now assume that our system is made of two superconductors at phase difference χ separated by a long central normal region C of length ℓ as sketched in Fig. (2). We assume that $\Delta(x)$ makes an abrupt transition from $\Delta e^{\pm i\chi/2}$ to 0 in the central region and $V(x)$ makes an abrupt transition from 0 in the leads to V_C in the central region. Here "abrupt" means rapid on the scale of ℓ. We assume $V_C < \mu$ so that the central region is metallic. The mass in C is written as m. The wave-functions in the central region, far from the interfaces, may be written $u(x) = C_1\exp(i\alpha_p x) + C_2\exp(-i\alpha_p x)$, $v(x) = C_3\exp(-i\alpha_h x) + C_4\exp(i\alpha_h x)$ with $\alpha_{p/h} = \{2m(\mu - V_C \pm E)\}^{\frac{1}{2}}$. We may define transmission matrices for the left and right interfaces, L and R by $\vec{C} = L\vec{A}^-$, $\vec{A}^+ = R\vec{C}$ in terms of which $M = R\cdot M^C\cdot L$, where M^C is the transmission matrix of C, given by the diagonal matrix with entries $[\exp(i\alpha_p\ell), \exp(-i\alpha_p\ell), \exp(-i\alpha_h\ell), \exp(i\alpha_h\ell)]$. For large ℓ, the integral along the imaginary E-axis is dominated by energies of $O(v_F/\ell)$ and the leading term in $1/\ell$ is determined by the L and R transmission matrices at $E = 0$. These in turn can be expressed in terms of the normal and Andreev reflection amplitudes at zero energy for particles and holes at the left-hand (L) and at the right-hand interface (R), $N_{L/R}^{p/h}$ and $A_{L/R}^{p/h}$. As a result, one obtains

$$I^{(0)}[\chi] = -\frac{4ev_F}{\pi\ell}\partial_\chi \vartheta^2(\chi) \quad , \tag{10}$$

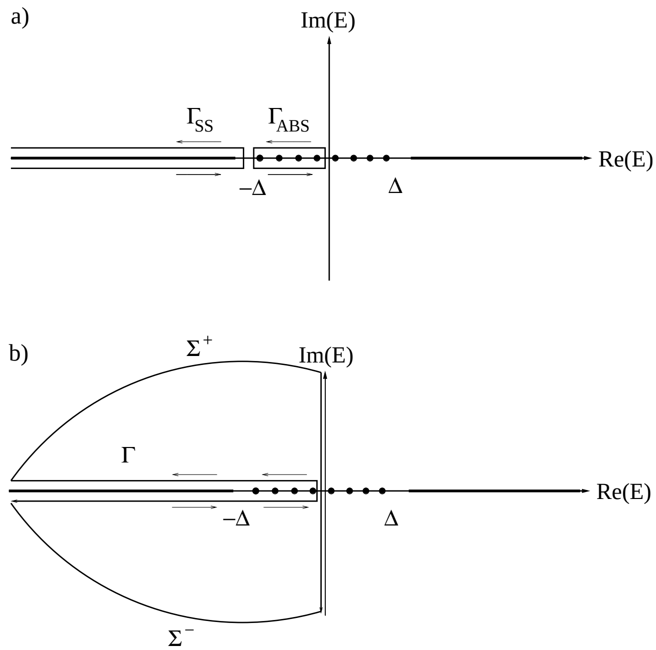

FIG. 1. Sketch of the deformation of the integration path Γ used in Eq.(9) to get to Eq.(10):
a) The integration path Γ sketched as $\Gamma = \Gamma_{\mathrm{ABS}} \cup \Gamma_{\mathrm{SS}}$, with Γ_{ABS} running around poles corresponding to ABS's and Γ_{SS} going around the energy interval corresponding to negative-energy SS's;
b) Adding the arcs Σ^+, Σ^-, whose contribution to the integral is zero as their radius is sent to ∞ (see text for the discussion), allows for trading the integral over Γ for an integral over the imaginary axis.

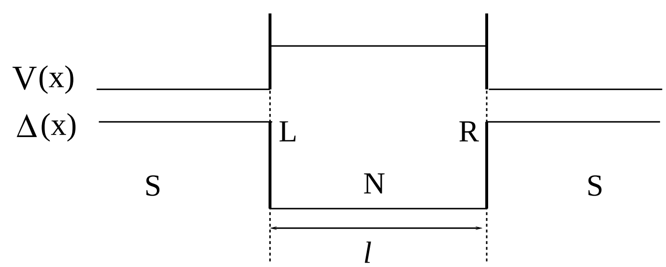

FIG. 2. Sketch $V(x), \Delta(x)$ in the SNS system. The L- and R-interfaces are assumed to be "sharp", that is, $V(x), \Delta(x)$ and the single-particle mass are assumed to vary over typical length scales $\ll \ell$. In this case scattering is localized at the interfaces and is fully encoded within the particle- and hole-normal- and Andreev- scattering amplitudes at the interfaces, $N_{L/R}^{p/h}, A_{L/R}^{p/h}$.

with $\vartheta(\chi) = \arccos\{\mathrm{Re}[N_R^p N_L^p e^{2i\alpha_F\ell} + A_R^p A_L^h e^{i\chi}]\}$, and α_F being the Fermi momentum. This leads to Ishi's sawtooth current [9] in the limit of perfect Andreev scattering at zero energy ($N_{L/R}^p = 0$, $|A_{L/R}^{p/h}| = 1$) and a complete suppression of the Josephson current when the Andreev reflection amplitude vanishes at either interface.

As an example, we explicitly compute $I^{(0)}[\chi]$ in a

model system whose S-N interfaces may be thought of as a generalization of the one studied in [10], that is, we assume that m is uniform, $\Delta(x)$ abruptly changes at the S-N interfaces and

$$V(x) = V_0[\delta(x) + \delta(x-\ell)] + V_C\theta(x)\theta(\ell - x) \quad (11)$$

where δ and θ are the Dirac delta-function and Heavyside step function respectively [11]. Eq. (10) now gives:

$$\vartheta(\chi) = \arccos\Bigg\{\mathrm{Re}\Bigg[-2\cos(\chi)\left[\frac{\alpha_F(\beta_F+\beta_F^*)}{\alpha_F^2+|\beta_F - iZ|^2}\right]^2 + \left(\frac{(\beta_F-\alpha_F-iZ)(\beta_F^*+\alpha_F+iZ)}{\alpha_F^2+|\beta_F-iZ|^2}\right)^2 e^{2i\alpha_F\ell}\Bigg]\Bigg\} \quad (12)$$

where $\alpha_F = \sqrt{2m(\mu - V_C)}$, $\beta_F = \{2m(\mu+i\Delta)\}^{\frac{1}{2}}$ and $Z = 2mV_0$. Note that two conditions must be satisfied for perfect Andreev reflection and hence a sawtooth current: $Z = \mathrm{Im}\beta_F = \sqrt{m(\sqrt{\mu^2+\Delta^2}-\mu)}$, $V_C = -Z^2/(2m)$. This is actually consistent with the results obtained in Ref.[4] within a tight-binding approach to the calculation of the dc Josephson current across an SNS system: getting perfect Andreev reflection requires, in general, fine-tuning two system parameters. On the other hand, allowing the central region to host an effectively attractive interaction between electrons should allow for the system to dynamically self-tune to the perfect Andreev reflection point, as ℓ becomes large [4]. Our general result, Eq.(10), also holds for tight-binding models where the spectrum may also include normal bound states. Indeed, for the nearest neighbor tight-binding model at half-filling studied in [4], one obtains $\vartheta(\chi) = \arccos\left[\frac{4\Delta_B^2\cos(\chi)+(1-\Delta_B^2)^2}{(1+\Delta_B^2)^2}\right]$, with the effective boundary coupling Δ_B defined as in [4]. Using this formula for $\vartheta(\chi)$, Eq.(10) gives the main formula for the dc Josephson current in Eq.(3.13) of [4], derived using a low energy effective boundary mode, taken in the symmetric case ($\Delta_L = \Delta_R$).

It is easy to generalize our contour result to compute the dc Josephson current at finite temperature T, $I[\chi;T]$. Now the contour Γ gets deformed into a sum of circles around the poles of the Fermi function at $\omega_n = 2\pi T(n+1/2)$, yielding

$$I[\chi;T] \approx 2eT\sum_{n=-\infty}^{\infty}\times \left[\frac{\partial_\chi\ \mathrm{Re}[A_R^pA_L^he^{i\chi}]}{\cosh\left(\frac{2\omega_n\ell}{v_F}\right) - \ \mathrm{Re}[N_R^pN_L^pe^{2i\alpha_F\ell}+A_R^pA_L^he^{i\chi}]}\right] \quad (13)$$

Of course, this gives our $T=0$ result at $T \ll v_F/\ell$ where we may approximate the sum by an integral. At $T \gg v_F/\ell$ we may approximate the sum by the two terms with $\omega_n = \pm\pi T$:

$$I[\chi;T] \approx 8eTe^{-2\pi T\ell/v_F}\partial_\chi\ \mathrm{Re}[A_R^pA_L^he^{i\chi}] + O\left(e^{-6\pi T\ell/v_F}\right). \quad (14)$$

This becomes exponentially small when $T \gg v_F/\ell$.

By using analytic properties of the scattering matrix for an SNS system, we have expressed the Josephson current, which has contributions from both bound and scattering states, as a single contour integral in the complex energy plane. In the limit of a long central region, we then proved that the current can be expressed in terms of properties of the Andreev bound states at the Fermi energy only: namely the normal and Andreev scattering amplitudes. The result holds at finite temperature provided that $T, v_F/\ell \ll \Delta$. This result shows that the Josephson current is a *universal* quantity, in this long length low temperature limit, and justifies the low energy Hamiltonian approach used in [4], which was crucial for treating Luttinger liquid interaction effects. It also paves the way towards an extension to the non-equilibrium (AC) Josephson effects.

Explicit derivation of Eq.(9) and of the formula for $\vartheta(\chi)$

The starting point to derive Eq.(9) and, specifically, the explicit formula for $\vartheta(\chi)$ in terms of the normal- and of the Andreev-reflection amplitudes at the Fermi level, is writing det$[S]$ (that is, $\mathcal{F}(E;\chi)$ and $\mathcal{G}(E;\chi)$) in terms of the R- and L-matrix elements. By definition of $\mathcal{F}$ and $\mathcal{G}$ and of the R- and L-transmission matrices, one finds that the following explicit formulas hold:

$$\begin{aligned}
&\mathcal{F}\,(E;\chi) = F_{0,0}(\chi;E) + F_{1,1}(E)e^{i[\alpha_p-\alpha_h]\ell} \\
&+ F_{-1,-1}(E)e^{-i[\alpha_p-\alpha_h]\ell} + F_{1,-1}(E)e^{i[\alpha_p+\alpha_h]\ell} \\
&+ F_{-1,1}(E)e^{-i[\alpha_p+\alpha_h]\ell} \\
&\mathcal{G}\,(E;\chi) = G_{0,0}(\chi;E) + G_{1,1}(E)e^{i[\alpha_p-\alpha_h]\ell} \\
&+ G_{-1,-1}(E)e^{-i[\alpha_p-\alpha_h]\ell} + G_{1,-1}(E)e^{i[\alpha_p+\alpha_h]\ell} \\
&+ G_{-1,1}(E)e^{-i[\alpha_p+\alpha_h]\ell},
\end{aligned} \quad (15)$$

with

$$\begin{aligned}
F_{0,0}(\chi) &= -(R_{1,1}R_{3,2} - R_{1,2}R_{3,1})(L_{1,3}L_{2,1} - L_{1,1}L_{2,3}) \\
&\quad - (R_{1,3}R_{3,4} - R_{1,4}R_{3,3})(L_{3,3}L_{4,1} - L_{3,1}L_{4,3}) \\
F_{1,1} &= (R_{1,1}R_{3,3} - R_{1,3}R_{3,1})(L_{1,1}L_{3,3} - L_{1,3}L_{3,1}) \\
F_{-1,-1} &= (R_{1,2}R_{3,4} - R_{1,4}R_{3,2})(L_{2,1}L_{4,3} - L_{4,1}L_{2,3}) \\
F_{1,-1} &= (R_{1,1}R_{3,4} - R_{3,1}R_{1,4})(L_{1,1}L_{4,3} - L_{1,3}L_{4,1}) \\
F_{-1,1} &= (R_{1,2}R_{3,3} - R_{3,2}R_{1,3})(L_{2,1}L_{3,3} - L_{2,3}L_{3,1})
\end{aligned} \quad (16)$$

and

$$\begin{aligned}
G_{0,0}(\chi) &= -(R_{2,1}R_{4,2} - R_{2,2}R_{4,1})(L_{1,4}L_{2,2} - L_{1,2}L_{2,4}) \\
&\quad - (R_{2,3}R_{4,4} - R_{2,4}R_{4,3})(L_{3,4}L_{4,2} - L_{3,2}L_{4,4}) \\
G_{1,1} &= (R_{2,1}R_{4,3} - R_{2,3}R_{4,1})(L_{1,2}L_{3,4} - L_{3,2}L_{1,4})
\end{aligned}$$

$$
\begin{aligned}
G_{-1,-1} &= (R_{2,2}R_{4,4} - R_{2,4}R_{4,2})(L_{2,2}L_{4,4} - L_{2,4}L_{4,2}) \\
G_{1,-1} &= (R_{2,1}R_{4,4} - R_{4,1}R_{2,4})(L_{1,2}L_{4,4} - L_{1,4}L_{4,2}) \\
G_{-1,1} &= (R_{2,2}R_{4,3} - R_{4,2}R_{2,3})(L_{2,2}L_{3,4} - L_{3,2}L_{2,4}) \qquad (17)
\end{aligned}
$$

and the explicit dependence upon χ set according to the definition of the R- and L-matrices. Using general properties of the transmission matrices, arising from the continuity equation for probability current, it is not difficult to show that Eqs.(16,17) are consistent with the identity $\mathcal{F}(E;\chi) = \mathcal{G}^*(E;\chi)$, that $\mathcal{G}(E;\chi)$ is real for real E and $-\Delta \leq E \leq \Delta$, and that the branch cuts of $\mathcal{G}(E;\chi)$ lie on the real axis, from $E \to -\infty$ to $E = -\Delta$ and from $E = \Delta$ to $E \to \infty$. From Eqs.(15), one then sees that

$$
\partial_\chi \ln \mathcal{G}(E;\chi) = \frac{\partial_\chi G_{0,0}(E;\chi)}{\mathcal{G}(E;\chi)} \quad . \qquad (18)
$$

Because, as E goes to infinity along a radius that does not overlap with the real axis, either the imaginary part of α_p, or the imaginary part of α_h, goes to $-\infty$, from Eq.(18) we find that $\partial_\chi \ln \mathcal{G}(E;\chi)$ exponentially vanishes as $|E| \to \infty$ off the real axis. Using Eq.(8) for the current and taking into account that, for large ℓ, contributions to the integral with $|\omega| \geq V$ are strongly suppressed, we may approximate α_p and α_h as

$$
\begin{aligned}
\alpha_p &\approx \alpha_F + i\omega\sqrt{\frac{m}{2(\mu - V_C)}} \\
\alpha_h &\approx \alpha_F - i\omega\sqrt{\frac{m}{2(\mu - V_C)}} \quad , \qquad (19)
\end{aligned}
$$

with $\alpha_F = \sqrt{2m(\mu - V_C)}$. Using Eqs.(19), the zero-temperature Josephson current to leading order in ℓ^{-1} is then given by

$$
\begin{aligned}
I^{(0)}[\chi] = -\frac{e}{\pi\ell}\sqrt{\frac{\mu - V_C}{2m}} &\int_{-\infty}^{\infty} dz\, \partial_\chi \bar{G}_{0,0}(\chi) \times \\
&\{ \bar{G}_{1,1}e^{-z} + \bar{G}_{-1,-1}e^{z} + \bar{G}_{1,-1}e^{2i\alpha_F\ell} \\
&+ \bar{G}_{-1,1}e^{-2i\alpha_F\ell} + \bar{G}_{0,0}(\chi)\}^{-1} \quad , \qquad (20)
\end{aligned}
$$

with the coefficients $\bar{G}_{a,b}$ being defined as the coefficients $G_{a,b}$ evaluated at $\omega = 0$, that is, setting $\alpha_p = \alpha_h = \alpha_F$, $\beta_p = \beta_h^* = \{2m_S[\mu + i\Delta]\}^{\frac{1}{2}}$. Computing the integral in Eq.(20), one eventually finds out

$$
I^{(0)}[\chi] = \frac{ev_F}{\pi\ell}\ \partial_\chi \ln^2\left[\frac{u_+(\chi)}{u_-(\chi)}\right] \quad , \qquad (21)
$$

with $v_F = \alpha_F/m$ and $u_\pm(\chi)$ being the roots of the second-degree equation

$$
\begin{aligned}
\bar{G}_{-1,-1}u^2 + [&\bar{G}_{1,-1}e^{2i\alpha_F\ell} + \bar{G}_{-1,1}e^{-2i\alpha_F\ell} \\
&+ \bar{G}_{0,0}(\chi)]u + \bar{G}_{1,1} = 0 \quad . \qquad (22)
\end{aligned}
$$

In order to prove that Eq.(21) yields Eq.(9) of the main text, we have to rewrite Eq.(22) in terms of the normal- and Andreev-scattering amplitudes at the Fermi level. To do so, we relate the scattering amplitudes at both interfaces to the R- and L-matrix elements. This may be readily done starting from the definition of the normal- and Andreev-scattering amplitudes. The result is

$$
\begin{aligned}
N_R^p(E) &= \frac{R_{2,4}R_{4,1} - R_{2,1}R_{4,4}}{R_{2,2}R_{4,4} - R_{2,4}R_{4,2}} \\
A_R^p(E) &= \frac{R_{2,1}R_{4,2} - R_{2,2}R_{4,1}}{R_{2,2}R_{4,4} - R_{2,4}R_{4,2}} \\
N_R^h(E) &= \frac{R_{2,3}R_{4,2} - R_{2,2}R_{4,3}}{R_{2,2}R_{4,4} - R_{2,4}R_{4,2}} \\
A_R^h(E) &= \frac{R_{2,4}R_{4,3} - R_{2,3}R_{4,4}}{R_{2,2}R_{4,4} - R_{2,4}R_{4,2}} \quad , \qquad (23)
\end{aligned}
$$

and

$$
\begin{aligned}
N_L^p(E) &= \frac{L_{1,2}L_{4,4} - L_{1,4}L_{4,2}}{L_{2,2}L_{4,4} - L_{2,4}L_{4,2}} \\
A_L^p(E) &= \frac{L_{3,2}L_{4,4} - L_{3,4}L_{4,2}}{L_{2,2}L_{4,4} - L_{2,4}L_{4,2}} \\
N_L^h(E) &= \frac{L_{2,2}L_{3,4} - L_{2,4}L_{3,2}}{L_{2,2}L_{4,4} - L_{2,4}L_{4,2}} \\
A_L^h(E) &= \frac{L_{1,4}L_{2,2} - L_{1,2}L_{2,4}}{L_{2,2}L_{4,4} - L_{2,4}L_{4,2}} \quad . \qquad (24)
\end{aligned}
$$

Using Eqs.(23,24) specified at $E = 0$, we may then rewrite Eq.(22) as

$$
u^2 + 1 - \{N_R^p N_L^p e^{2i\alpha_F\ell} + A_R^p A_L^h e^{i\chi} + \text{c.c.}\}u = 0 \quad , \qquad (25)
$$

with all the scattering amplitudes in Eq.(25) evaluated at $E = 0$.

Putting together Eqs.(21,25), one readily derives Eq.(9) and the related formula for $\vartheta(\chi)$.

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